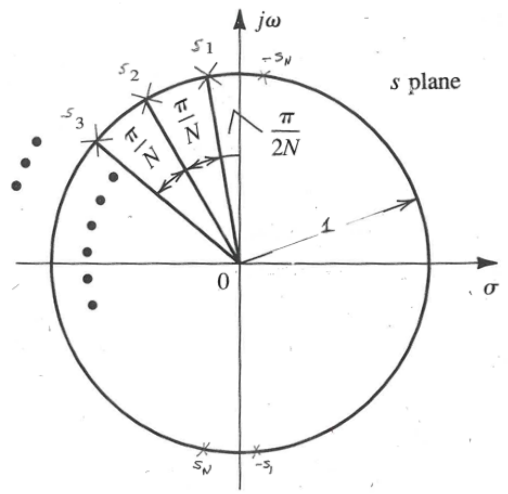
Butterworth Filters

Butterworth filters, often called maximally flat filters, achieve a frequency response of the form:

$$|H(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_b}\right)^{2N}}}$$

where $N$ can be any positive integer. To see how this can be achieved, we first consider the related transfer function

$$\bar{H}(s) = \frac{1}{1+(-1)^N s^{2N}} \xrightarrow{s=j\omega} \frac{1}{1+(\omega)^{2N}}$$




Since the denominator $\bar{H}(s)$ is a 2*Nth order polynomial, the root (poles) satisfy $(-1)^N s^{2N} = -1$. These roots can be expressed as:

$s_m = \pm e^{j\phi_m}$ where $\phi_m = \frac{\pi}{2} + \frac{\pi}{2N}(1+2m)$ and $m = 1,2 \ldots N$. These poles are shown in the figure below. Note that the poles $s_m = e^{j\phi_m}$ are in the left-half plane, and poles $s_m = -e^{j\phi_m}$ are in the right-half plane. Hence, $\bar{H}(s)$ can be written in the form:

$$\bar{H}(s) = \frac{1}{\underbrace{(s-s_1)(s-s_2)\ldots(s-s_N)}_{\text{Left Plane poles}} * \underbrace{(s-(-s_1))(s-(-s_2))\ldots(s-(-s_N))}_{\text{Right Plane poles}}} = \frac{1}{1+(-1)^N s^{2N}} \xrightarrow{s=j\omega} \frac{1}{1+(\omega)^{2N}}$$

$\bar{H}(s)$ certainly, has the maximally flat rolloff that we want with $\omega$, but has the problem of having RHP poles, which would make it unstable. But when $s = j\omega$,

$$|\bar{H}(j\omega)| = \frac{1}{|j\omega - s_1||j\omega - s_2|\ldots|j\omega - s_N| * |j\omega + s_1||j\omega + s_2|\ldots|j\omega + s_N|} = \left[\frac{1}{|j\omega - s_1||j\omega - s_2|\ldots|j\omega - s_N|}\right]^2$$

Which shows that the contribution of the LHP and RHP poles are the same when $s = j\omega$, so a transfer function with just the LHP poles would have a response:

$$H(s) = \frac{1}{(s-s_1)(s-s_2)\ldots(s-s_N)} \xrightarrow{s=j\omega} \frac{1}{\sqrt{1+\omega^{2N}}}$$

The polynomial $(s-s_1)(s-s_2)...(s-s_N)$ where the roots are given by the expression above are called Butterworth Poynomials. Recognizing that all the poles but one (when N is odd) are complex conjugate pairs, these polynomials can be expressed in factored form as:

| n (order) | Normalized Denominator Polynomials in Factored Form |
|---|---|
| 1 | (1+s) |
| 2 | $(1+1.414s+s^2)$ |
| 3 | $(1+s)(1+s+s^2)$ |
| 4 | $(1+0.765s+s^2)(1+1.848s+s^2)$ |
| 5 | $(1+s)(1+0.618s+s^2)(1+1.618s+s^2)$ |
| 6 | $(1+0.518s+s^2)(1+1.414s+s^2)(1+1.932s+s^2)$ |
| 7 | $(1+s)(1+0.445s+s^2)(1+1.247s+s^2)(1+1.802s+s^2)$ |
| 8 | $(1+0.390s+s^2)(1+1.111s+s^2)(1+1.663s+s^2)(1+1.962s+s^2)$ |
| 9 | $(1+s)(1+0.347s+s^2)(1+s+s^2)(1+1.532s+s^2)(1+1.879s+s^2)$ |
| 10 | $(1+0.313s+s^2)(1+0.908s+s^2)(1+1.414s+s^2)(1+1.782s+s^2)(1+1.975s+s^2)$ |

Finally, for an arbitrary break frequency $\omega_b$, the Butterworth transform function would be

$$H(s)=\frac{1}{(\frac{s}{\omega_b}-s_1)(\frac{s}{\omega_b}-s_2)...(\frac{s}{\omega_b}-s_N)}\xrightarrow{s=j\omega}\frac{1}{\sqrt{1+(\frac{\omega}{\omega_b})^{2N}}}$$